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# Tate classes on a product of two Picard modular surfaces <sup>☆</sup>

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## Abstract

We compute the space of codimension 2 Tate classes on a product of two Picard modular surfaces in terms of automorphic representations on  $GL(n)$ ,  $n < 4$ . The relevant part of the fourth cohomology splits into subspaces indexed by pairs of such automorphic representations. When these representations are not automorphically induced, the corresponding Tate classes are shown to be abelian.

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Let  $X$  be a smooth projective variety defined over a number field  $E$ , and let

$$\bar{X} = X \times_E \overline{\mathbb{Q}}.$$

For a prime  $\ell$ , let  $H^i(X) = H^i(\bar{X}_{\text{ét}}, \mathbb{Q}_\ell)$  be the  $\ell$ -adic cohomology of  $X$ . The Galois group  $G = \text{Gal}(\overline{\mathbb{Q}}/E)$  acts on  $H^i(X)$  by a representation  $\rho_i$ . For any  $j \in \mathbb{Z}$ , let  $H^i(X)(j)$  denote the representation of  $G$  on  $H^i(X)$  defined by  $\rho_i \otimes \chi_\ell^j$ , where  $\chi_\ell$  is the cyclotomic character. For any finite extension  $L/E$ , define

$$G_L = \text{Gal}(\overline{\mathbb{Q}}/L),$$

which is an open subgroup of  $G$ . A *Tate class* is an element of  $H^{2i}(X)(i)^{G_L}$  for some  $L/E$ . A Tate class is *defined over  $L$*  if it is fixed by  $G_L$ .

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To each cycle  $Z$  on  $X$  of codimension  $i$  and defined over  $L$  there is associated a cohomology class  $c(Z) \in H^{2i}(X)(i)$  by  $\ell$ -adic Poincaré duality. Such a cohomology class is said to be *algebraic*. In this situation  $c(Z)$  is fixed by  $G_L$ . Tate's conjecture asserts that conversely every Tate class is algebraic.

A basic first step in studying Tate's conjecture is to identify the Tate classes on  $X$ . This is the goal of the present paper, when  $X$  is a product of two Picard modular surfaces (relative to a fixed imaginary quadratic field), and  $i = 2$ . We shall determine the Tate classes in terms of the automorphic representations associated to the two surfaces.

In [MP], Murty and Prasad studied the same problem for a product of Hilbert modular surfaces. Their work applies in addition to the case of two modular surfaces relative to two distinct real quadratic fields.

*Notation:*  $\mathbf{A}$  denotes the ring of adeles of  $\mathbf{Q}$ , and  $\mathbf{A}_f$  the finite adeles. If  $E$  is a number field, then  $\mathbf{A}_E$  denotes the adeles of  $E$ . If  $\pi$  is an automorphic representation then  $\pi_f = \otimes_{p < \infty} \pi_p$ , and  $\pi_E$  denotes the base change to  $E$ .

## 1. Review of the cohomology of a Picard surface

We collect some facts from [LR], where full details can be found. Fix a quadratic imaginary extension  $E/\mathbf{Q}$ , and a Hermitian inner product on  $E^3$  of signature  $(2, 1)$ . Let  $\mathrm{GU}$  be the associated quasi-split unitary similitude group over  $\mathbf{Q}$ .  $\mathrm{GU}_\infty = \mathrm{GU}(\mathbf{R})$  is the real Lie group  $\mathrm{GU}(2, 1)$ . The symmetric space associated to  $\mathrm{GU}$  is

$$B = \mathrm{GU}(\mathbf{R})/K_\infty Z_\infty,$$

where  $K_\infty \subset \mathrm{GU}(\mathbf{R})$  is a maximal compact subgroup, and  $Z$  is the center of  $\mathrm{GU}$ .  $B$  can be identified in a natural way with the unit ball in  $\mathbf{C}^2$ .

Fix a nontrivial algebraic homomorphism  $h: \mathrm{Res}_{\mathbf{R}}^{\mathbf{C}}(\mathbf{G}_m) \rightarrow \mathrm{GU}(\mathbf{R})$  which satisfies the axioms for defining a Shimura variety. Fix an open compact subgroup  $K$  of  $\mathrm{GU}(\mathbf{A}_f)$  and let  $S_K = S_K(\mathrm{GU}, h)$  be the associated (compactified) Shimura variety.  $S_K$  is defined over  $E$  and  $S_K(\mathbf{C})$  is the compactification of

$$\mathrm{GU}(\mathbf{Q}) \backslash (B \times \mathrm{GU}(\mathbf{A}_f)) / K,$$

which is a disjoint union of arithmetic quotients of  $B$ .

Let  $\mathcal{H} = \mathcal{H}_K(\mathbf{Q})$  be the Hecke algebra of  $\mathbf{Q}$ -valued compactly supported bi- $K$ -invariant functions on  $\mathrm{GU}(\mathbf{A}_f)$ . For any  $\mathbf{Q}$ -algebra  $A$ , let  $\mathcal{H}(A) = \mathcal{H} \otimes A$ . The Galois group

$$G = \mathrm{Gal}(\overline{\mathbf{Q}}/E)$$

and the Hecke algebra  $\mathcal{H}$  both act on  $H^*(S_K)$ . Over  $\overline{\mathbf{Q}}_\ell$ , the degree 2 cohomology of  $S$  decomposes as an  $\mathcal{H}(\overline{\mathbf{Q}}_\ell)$ -module into isotypic components which are stable under the commuting action of  $G$ :

$$H^2(S_K) \otimes \overline{\mathbf{Q}}_\ell = \bigoplus_{\pi_f} H(\pi_f). \quad (1)$$

Here  $\pi$  runs through the automorphic representations of  $\mathrm{GU}(\mathbf{A})$  for which

- $\pi$  occurs in the discrete spectrum of  $\mathrm{GU}(\mathbf{A})$ .
- $\pi_\infty \in \{triv, D^+, D, D^-\}$ , where  $triv$  is the trivial representation, and  $D^+, D, D^-$  are the lowest weight holomorphic, non-holomorphic, and anti-holomorphic discrete series representations of  $\mathrm{GU}_\infty$  with trivial central characters.
- The subspace  $\pi_f^K(\mathbf{C})$  of  $\pi_f$  consisting of  $K$ -fixed vectors is nonzero (and hence is an irreducible finite-dimensional representation of  $\mathcal{H}$ ).

Any such  $\pi$  is either one-dimensional or cuspidal and infinite-dimensional. The second condition is equivalent to the property that

$$H^2(\mathrm{Lie}(\mathrm{GU}_\infty), K_\infty', \pi_\infty) \neq 0,$$

where  $K_\infty'$  is the centralizer of the center of  $K_\infty$  in  $\mathrm{GU}_\infty$ . There are finitely many  $\pi$  which satisfy the conditions.

There is a (noncanonical) decomposition

$$H(\pi_f) = V_{\pi_f} \otimes \pi_f^K,$$

where  $V_{\pi_f}$  is a  $\overline{\mathbf{Q}}_\ell$ -vector space of dimension  $d(\pi_f) \leq 3$ , and  $\pi_f^K = \pi_f^K(\overline{\mathbf{Q}}) \otimes \overline{\mathbf{Q}}_\ell$ , where  $\pi_f^K(\overline{\mathbf{Q}})$  is a  $\overline{\mathbf{Q}}$ -form of  $\pi_f^K(\mathbf{C})$ .  $G$  acts continuously on  $H(\pi_f)$  by a representation of the form  $\rho_{\pi_f} \otimes 1$ . (This is the definition of  $\rho_{\pi_f}$ ). The isomorphism class of  $\rho$  is independent of the choice of the above decomposition; for concreteness, one can take  $V_{\pi_f} = \mathrm{Hom}_{\mathcal{H}}(\pi_f^K, H^2(S_K) \otimes \overline{\mathbf{Q}}_\ell)$ , but this obscures the point of view that  $\rho_{\pi_f}$  acts on the cohomology. Fix an embedding

$$\varepsilon : \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}.$$

The representation  $\rho$  is unramified at almost every place  $w$  of  $E$ . The local  $L$ -factor of  $\rho$  at such a place

$$L_w(s, \rho) = \det(1 - \varepsilon(\rho(\mathrm{Fr}_w))q_w^{-s})^{-1},$$

where  $\mathrm{Fr}_w \in G$  is a geometric Frobenius element.

The relationship between  $\rho_{\pi_f}$  and  $\pi_f$  is the following. There is an automorphic representation  $\sigma_\pi$  of  $\mathrm{GL}_{d(\pi_f)}(\mathbf{A}_E)$  associated to  $\pi$  in a natural way (see below) such that for almost all  $w$

$$L_w(s, \rho_{\pi_f}) = L_w(s - 1, \sigma_\pi). \quad (2)$$

The dimension  $d(\pi_f)$  of  $\rho_{\pi_f}$  is the number of  $\pi_\infty$  such that  $\pi_f \otimes \pi_\infty$  occurs in the discrete spectrum. This number varies according to the classification of  $\pi_f$  as stable, endoscopic, or 1-dimensional. If  $\pi_f$  is stable and infinite-dimensional,

then  $d(\pi_f) = 3$ , and

$$\sigma_\pi = (\pi_0)_E \otimes \bar{\chi}_\pi,$$

where  $\pi_0 = \pi|_{U(3)}$ . (This is independent of the choice of  $\pi_\infty$ ). In the endoscopic cases,  $d(\pi_f) = 1$  or  $2$ . The 2-dimensional case occurs in certain instances when  $\pi_0$  is an endoscopic lift of some  $\tau_1 \otimes \tau_2$  on  $U(2) \times U(1)$ . In these cases,

$$\sigma_\pi = (\tau_1)_E \otimes \bar{\chi}_\pi,$$

where  $(\tau_1)_E$  is the nonstandard base change of  $\tau_1$ . The remaining (one-dimensional) cases are summarized in [BR].

Note the difference between  $\sigma_\pi$  and  $\pi_E$  in the stable case:

$$\pi_E \cong (\pi_0)_E \otimes \bar{\chi}_\pi$$

is a representation of  $GL_3(\mathbf{A}_E) \times GL_1(\mathbf{A}_E)$ , while

$$\sigma_\pi = (\pi_0)_E \otimes (\bar{\chi}_\pi \circ \det)$$

is a representation of  $GL_3(\mathbf{A}_E)$ . It is therefore conceivable that  $\sigma_{\pi_1} \cong \sigma_{\pi_2}$  when  $(\pi_1)_E \not\cong (\pi_2)_E$ , i.e. two distinct  $L$ -packets on  $GU$  could give rise to isomorphic Galois representations. In this case however,  $(\pi_1)_0$  and  $(\pi_2)_0$  belong to the same  $L$ -packet on  $U$ .

**Definition 1.** We say that  $\pi$  is **AI** if  $\sigma_\pi$  is automorphically induced from a Hecke character of some field  $L$  of degree  $d(\pi_f)$  over  $E$ .

In the stable case,  $\pi$  is AI if and only if  $(\pi_0)_E$  is automorphically induced. Also note that if  $d(\pi_f) = 1$  then  $\pi_f$  is (trivially) AI by this definition.

We recall two theorems from [BR].

**Theorem 1** (Blasius and Rogawski [BR, 2.2.1]). *Let  $\pi$  and  $\rho = \rho_{\pi_f}$  be as above. For any number field  $L/E$  let  $\rho_L = \rho|_{G_L}$ . Then one of the following two statements holds:*

- (1)  $\rho_L$  is irreducible for every finite extension  $L/E$ .
- (2) *There exists an extension  $L/E$  of degree  $d(\pi_f)$  and an algebraic Hecke character  $\Psi$  of  $L$  such that  $\rho \cong \text{Ind}_L^E(\Psi)$ .*

The second case occurs precisely when  $\pi$  is AI.

This irreducibility result is used in [BR] to prove the algebraicity of the Tate classes in  $H^2(S_K)(1)$ . For a large class of  $\pi_f$ , there are no associated Tate classes:

**Theorem 2** (Blasius and Rogawski [BR, 3.2.1]). *Let  $H^T(\pi_f)$  denote the space of Tate classes in  $H(\pi_f)(1)$  for  $S_K$ . Then*

$$H^T(\pi_f) = \begin{cases} H(\pi_f)(1) & \text{if } d(\pi_f) = 1 \text{ and } \text{Inf}(\pi_f) = D \text{ or triv} \\ \{0\} & \text{otherwise.} \end{cases}$$

*All such Tate classes are defined over abelian extensions of  $E$ .*

For use in the next section, we record the following observation.

**Lemma 1.** *Suppose  $\pi_f$  is as above. Then  $V_{\pi_f}^* = V_{\pi_f^*}(2)$ .*

**Proof.** Let  $\sigma = \sigma_\pi$  as in (2). Given a place  $w$  of  $E$ , let  $\text{Fr}_w$  be a geometric Frobenius element of  $\text{Gal}(\overline{\mathbb{Q}}/E)$ , and let  $g_w(\sigma)$  be the Langlands class of  $\sigma$  at  $w$ . Then (2) is equivalent to

$$\rho_\pi(\text{Fr}_w) \sim q_w g_w(\sigma),$$

for almost all  $w$ . Then

$$\begin{aligned} \rho_\pi^*(\text{Fr}_w) &\sim q_w^{-1} g_w(\sigma)^{-1} \\ &= q_w^{-2} (q_w g_w(\sigma)^{-1}) \\ &= q_w^{-2} \rho_{\pi^*}(\text{Fr}_w), \end{aligned}$$

since  $\sigma_\pi^* = \sigma_{\pi^*}$ . Therefore  $\rho_\pi^* \cong \rho_{\pi^*} \otimes \chi_\ell^2$  by Cebotarev and continuity of the  $\rho$ 's.  $\square$

## 2. Cohomology of a product of Picard surfaces

If  $S_1$  and  $S_2$  are two Picard surfaces for open compact subgroups  $K_1$  and  $K_2$ , then by the Künneth formula we have

$$H^4(S_1 \times S_2)(2) = \bigoplus_{i+j=4} H^i(S_1) \otimes H^j(S_2)(2).$$

The most interesting part of this decomposition is

$$H^2(S_1)(1) \otimes H^2(S_2)(1).$$

(Although the other summands will provide Tate classes, the Galois representations on  $H^j$ ,  $j \neq 2$ , are abelian ([Ro2, Section 4.3]), and indeed in the nontrivial cases ( $j = 1, 3$ ) are attached to the Albanese varieties, which are of CM-type and for which Tate's conjecture is known.)

Suppose  $R$  is a ring, and that  $V, W$  are free  $R$ -modules on which  $G$  acts continuously. Recall that

$$V^* \otimes W \cong \text{Hom}_R(V, W)$$

as representations, where  $G$  acts on the right by  $(gf)(v) = g(f(g^{-1}v))$ . It follows from this that  $f \in \text{Hom}_R(V, W)^G$  if and only if  $f \in \text{Hom}_{R[G]}(V, W)$ , i.e. if and only if  $f$  is an intertwining operator from  $V$  to  $W$ .

In order to use automorphic representations to study the Tate classes in  $H^2(S_1)(1) \otimes H^2(S_2)(1)$ , we extend scalars to  $\overline{\mathbf{Q}}_\ell$ :

$$\begin{aligned} & [H^2(S_1)(1) \otimes H^2(S_2)(1)] \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell \\ &= [H^2(S_1)(1) \otimes \overline{\mathbf{Q}}_\ell] \otimes [H^2(S_2)(1) \otimes \overline{\mathbf{Q}}_\ell] \\ &= \left( \bigoplus_i V_{\pi_i}(1) \otimes \pi_i^{K_1} \right) \otimes \left( \bigoplus_j V_{\pi_j}(1) \otimes \pi_j^{K_2} \right), \end{aligned}$$

where  $\pi_i$  and  $\pi_j$  are representations of  $\mathrm{GU}(\mathbf{A}_f)$ , and  $V_{\pi_i}$  is a  $\overline{\mathbf{Q}}_\ell$ -vector space of dimension  $\leq 3$ . We rearrange the factors and pull out the sums to get:

$$\bigoplus_{i,j} (V_{\pi_i}(1) \otimes V_{\pi_j}(1)) \otimes (\pi_i^{K_1} \otimes \pi_j^{K_2}).$$

$G$  acts on each summand above by  $(\rho_i(1) \otimes \rho_j(1)) \otimes 1$ .

For a number field  $L$  containing  $E$  we need to compute the  $G_L$ -invariant subspace of  $V_{\pi_1}(1) \otimes V_{\pi_2}(1)$ . This is isomorphic to the space

$$\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[G_L]}(V_{\pi_1}(1)^*, V_{\pi_2}(1)) = \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[G_L]}(V_{\pi_1}^*(-1), V_{\pi_2}(1)).$$

By Lemma 1,  $V_{\pi_1}^* = V_{\pi_1}(2)$ , and for notational convenience we replace  $\pi_1$  with  $\pi_1^*$ . Thus we must determine

$$\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[G_L]}(V_{\pi_1}(1), V_{\pi_2}(1))$$

for various  $\pi_i$ . This space is canonically isomorphic to

$$\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell[G_L]}(V_{\pi_1}, V_{\pi_2}).$$

**Lemma 2.** *Let  $(\sigma, V)$  and  $(\tau, W)$  be  $n$ -dimensional representations of  $G$  over  $\overline{\mathbf{Q}}_\ell$ . Suppose  $H$  is an open normal subgroup of  $G$ , and  $\tau|_H$  is irreducible. Then  $\sigma|_H \cong \tau|_H$  iff  $\sigma \cong \tau \otimes \phi$ , for some  $\phi : G \rightarrow \overline{\mathbf{Q}}_\ell^*$ , trivial on  $H$ .*

**Proof.** Fixing bases for  $V$  and  $W$ , we view  $\sigma$  and  $\tau$  as maps from  $G$  into  $\mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ . Suppose there exists  $A \in \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$  such that

$$\sigma(h) = A\tau(h)A^{-1}$$

for all  $h \in H$ . For any  $g \in G$ , define

$$\phi(g) = \tau(g)^{-1}A^{-1}\sigma(g)A \in \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell).$$

Clearly  $\phi(h) = 1$  for all  $h \in H$ . In fact  $\phi(g)$  is a scalar for all  $g \in G$ . This follows because one computes directly that

$$\phi(g)^{-1}\tau(h)\phi(g) = \tau(h) \quad \text{for all } g \in G, h \in H,$$

using the fact that  $H$  is normal in  $G$ . Because  $\tau|_H$  is irreducible, Schur's lemma implies that  $\phi(g) \in \overline{\mathbf{Q}}_\ell^*$ . Thus  $\sigma \cong \tau \otimes \phi$ . The converse is clear.  $\square$

**Lemma 3.** *Suppose either  $\pi_1$  or  $\pi_2$  is non-AI. Let  $\sigma_i = \sigma_{\pi_i}$  as in (2). Then for any finite Galois extension  $L/E$ ,*

$$\text{Hom}_{\overline{\mathbf{Q}}_\ell[G_L]}(V_{\pi_1}, V_{\pi_2}) = \begin{cases} \overline{\mathbf{Q}}_\ell & \text{if } \sigma_1 \cong \sigma_2 \otimes \phi \text{ for some character } \phi \\ & \text{of } \mathbf{A}_E^* \text{ which is trivial on } E^* N_E^L(\mathbf{A}_L^*). \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** If for example  $\pi_1$  is non-AI, then  $(V_{\pi_1}, \rho_{\pi_1 L})$  is an irreducible representation of  $G_L$  of dimension 2 or 3. Then the  $\overline{\mathbf{Q}}_\ell$ -dimension of the above space is the multiplicity of  $\rho_{\pi_1 L}$  in  $\rho_{\pi_2 L}$  by Schur's lemma. Because the dimension of  $\rho_{\pi_2 L}$  is  $\leq 3$ , this multiplicity can only be 0 or 1. In the latter case,  $\pi_2$  must also be non-AI, and  $\rho_{\pi_1 L} \cong \rho_{\pi_2 L}$ , which by the previous lemma occurs if and only if

$$\rho_{\pi_1} \cong \rho_{\pi_2} \otimes \phi,$$

for some character  $\phi : G \rightarrow \overline{\mathbf{Q}}_\ell^*$ , trivial on  $G_L$ . By the relationship between  $\sigma_i$  and  $\rho_{\pi_i}$  and strong multiplicity one for  $\text{GL}(n)$ , this is equivalent to  $\sigma_1 \cong \sigma_2 \otimes \phi$ , where we identify  $\phi$  with the character of  $\mathbf{A}_E^*$  obtained by pulling back the Artin map:

$$\mathbf{A}_E^*/E^* \rightarrow \text{Gal}(L/E)^{ab} \xrightarrow{\phi} \overline{\mathbf{Q}}_\ell^* \xrightarrow{\varepsilon} \mathbf{C}^*. \quad \square$$

We say that a Tate class in  $H^2(S_1)(1) \otimes H^2(S_2)(1)$  is a *new Tate class* if it is not a tensor product of Tate classes on the two factors. Because the Tate conjecture is known for each surface, the Tate conjecture for codimension-2 cycles on  $S_1 \times S_2$  depends on finding algebraic cycles for the new Tate classes.

In light of Theorem 2, the above lemma immediately implies the following:

**Theorem 3.** *Let  $\pi_1$  and  $\pi_2$  be cuspidal and cohomological for  $S_1$  and  $S_2$ , respectively. Then:*

- (1) *If exactly one of  $\pi_1, \pi_2$  is AI, then  $H(\pi_1)(1) \otimes H(\pi_2)(1)$  contains no Tate classes.*
- (2) *If  $\pi_1$  and  $\pi_2$  are both non-AI, then  $H(\pi_1)(1) \otimes H(\pi_2)(1)$  contains a Tate class if and only if  $\sigma_1^* \cong \sigma_2 \otimes \phi$  for some finite order Hecke character  $\phi$  of  $E$ . In this case, the subspace of Tate classes has the same dimension as  $\pi_1^{K_1} \otimes \pi_2^{K_2}$ , and all such Tate classes are new and defined over the abelian extension of  $E$  defined by  $\phi$ .*

### 3. Tate classes in the AI case

It remains to consider the cases where  $\pi_1$  and  $\pi_2$  are both AI. By symmetry, we can assume  $d(\pi_1) \leq d(\pi_2) \leq 3$ . Thus there are six cases, which we refer to as  $(d(\pi_1), d(\pi_2))$ . The cases where  $d(\pi_2) = 3$  are complicated by the possibility that  $\sigma_{\pi_2}$  (or  $\sigma_{\pi_1}$ ) could be AI from a non-normal cubic extension.

As could be expected, the new Tate classes coming from AI representations are generally *not* defined over abelian extensions of  $E$ .

The (1,1) case can be handled exactly as in the non-AI case, and the second statement of the above theorem applies here to characterize the Tate classes, with the exception that these may be old Tate classes. (By Theorem 2, this is the only case which contributes old Tate classes).

For the general case, let  $M/E$  be a field extension of degree  $\leq 3$ , and let  $\phi : G_M \rightarrow \overline{\mathbf{Q}}_\ell^*$  be a continuous character. Let  $\rho = \text{Ind}_M^E(\phi)$ . We review the fact that if  $\phi$  is replaced by any of its Galois conjugates, the resulting induced representation is isomorphic to  $\rho$ . First suppose  $M/E$  is Galois, and let  $\tau \in G - G_M$ . Then

$$\rho_M = \begin{cases} \phi \oplus \phi^\tau & \text{if } [E : M] = 2 \\ \phi \oplus \phi^\tau \oplus \phi^{\tau^2} & \text{if } [E : M] = 3, \end{cases}$$

where  $\phi^\tau(\sigma) = \phi(\tau^{-1}\sigma\tau)$ . Any of the characters  $\phi, \phi^\tau, \phi^{\tau^2}$  can be used to define  $\rho$ , i.e.

$$\rho = \text{Ind}_M^E(\phi) \cong \text{Ind}_M^E(\phi^\tau) \cong \text{Ind}_M^E(\phi^{\tau^2}).$$

The same statement is true in the case where  $M/E$  is a non-normal cubic extension. In this case, let  $\tilde{M}$  be the Galois closure of  $M$  over  $E$ , so that  $G/G_{\tilde{M}} \cong S_3$ . Let  $\tau \in G$  be an element which has order 3 in this quotient. Then

$$\rho_{\tilde{M}} = \phi \oplus \phi^\tau \oplus \phi^{\tau^2}.$$

Here the characters on the right are characters of  $G_{\tilde{M}}$ ; although  $\phi$  extends to  $G_M$ , this is not the case for  $\phi^\tau$  and  $\phi^{\tau^2}$  since  $G_M$  is not a normal subgroup of  $G$ . However it is easy to see that  $\phi^\tau$  extends to  $G_{\tau(M)} = \tau G_M \tau^{-1}$ , and  $\phi^{\tau^2}$  extends to  $G_{\tau^2(M)}$ . In this way we have

$$\rho = \text{Ind}_M^E(\phi) \cong \text{Ind}_{\tau(M)}^E(\phi^\tau) \cong \text{Ind}_{\tau^2(M)}^E(\phi^{\tau^2}).$$

**Theorem 4.** *Suppose  $\pi_1$  and  $\pi_2$  are both AI, so that  $\sigma_i = \text{AI}_{M_i}^E(\phi_i)$  for algebraic Hecke characters  $\phi_i$  of fields  $M_i$  of degree  $d(\pi_i)$  over  $E$ . Let  $\tilde{M}$  be the normal closure of  $M_1 M_2$  over  $E$ . Then  $H(\pi_1)(1) \otimes H(\pi_2)(1)$  contains a Tate class if and only if*

$$(\phi_2)_{\tilde{M}}^{-1} = (\phi_1)_{\tilde{M}} \otimes \theta$$

for some finite order Hecke character  $\theta$  of  $\tilde{M}$ . (The subscript  $\tilde{M}$  denotes base change). The corresponding Tate classes are defined over the smallest field  $L$  on which  $\theta$  is trivial.

**Proof.** Identifying  $\phi_i$  with Galois characters by class field theory, we have  $\rho_1 = \text{Ind}_{M_1}^E(\phi_1)$  and  $\rho_2 = \text{Ind}_{M_2}^E(\phi_2)$ . Suppose  $\text{Hom}_L(\rho_1, \rho_2) \neq 0$ . Enlarging  $L$  if necessary, we may assume  $\tilde{M} \subset L$  and that  $L/\tilde{M}$  is a normal extension. Then as representations of  $G_L$ ,  $\rho_1$  and  $\rho_2$  split into sums of characters. At least one of the characters coming from  $\rho_{1L}$  must equal one of those coming from  $\rho_{2L}$ . By the preceding discussion, we do not lose generality if we assume

$$\phi_{1L} = \phi_{2L}.$$

Then by Lemma 2,

$$(\phi_2)_{\tilde{M}} = (\phi_1)_{\tilde{M}} \otimes \theta,$$

for some character  $\theta$  of  $G_{\tilde{M}}$  which is trivial on  $G_L$ . This is equivalent to the assertion in the theorem (replacing  $\rho_1$  by  $\rho_1^*$  amounts to replacing  $\phi_1$  by  $\phi_1^{-1}$ ).

Conversely, given that  $(\phi_2)_{\tilde{M}} = (\phi_1)_{\tilde{M}} \otimes \theta$ , it is easy to see that there is a nonzero intertwining operator from  $\rho_{1L}$  to  $\rho_{2L}$  for any  $L$  containing  $\tilde{M}$  and on which  $\theta$  is trivial.  $\square$

When  $\rho_1$  is one-dimensional, we can refine the above condition:

**Theorem 5.** Suppose  $d(\pi_1) = 1$  and  $\pi_2$  is AI. Then  $H(\pi_1)(1) \otimes H(\pi_2)(1)$  contains a new Tate class if and only if

$$\sigma_2 = AI_M^E((\sigma_1^{-1})_M \otimes \theta)$$

for some finite order Hecke character  $\theta$  of  $M$ .

**Proof.** We will assume  $d(\pi_2) = 3$  and that  $\sigma_2$  is AI from a non-normal cubic extension  $M/E$ , the other cases being easier. Thus let  $\phi: G_M \rightarrow \overline{\mathbb{Q}}_\ell^*$  be an algebraic Hecke character, and let  $\rho_2 = \text{Ind}_M^E(\phi)$ . Suppose  $\text{Hom}_L(\rho_1, \rho_2) \neq 0$ . Let  $\tilde{M}$  be the normal closure of  $M$  over  $E$ . Enlarging  $L$ , we assume that  $\tilde{M} \subset L$  is a normal extension. Then as in the proof of the above theorem, we have

$$\rho_{1\tilde{M}} = \phi_{\tilde{M}} = \phi \otimes \theta^{-1}$$

for some finite order character  $\theta$  of  $G_{\tilde{M}}$ . Extend  $\theta$  to a character of  $G_M$  by the formula

$$\theta(\sigma) = \phi(\sigma)\rho_1(\sigma^{-1}), \quad \sigma \in G_M.$$

Then  $\rho_{1M} = \phi \otimes \theta^{-1}$ , so

$$\rho_2 = \text{Ind}_M^E(\rho_{1M} \otimes \theta),$$

which gives the desired formula. The converse is clear.  $\square$

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